Action of special linear groups to the tensor of indeterminates, classical invariants of binary forms and hyperdeterminant*

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Abstract

In this paper, we study the ring of invariants under the action of $\mathrm{SL}(m,K)\times\mathrm{SL}(n,K)$ and $\mathrm{SL}(m,K)\times\mathrm{SL}(n,K)\times\mathrm{SL}(2,K)$ on the 3-dimensional array of indeterminates of form $m\times n\times 2$, where K is an infinite field. And we show that if $m=n\geq 2$, then the ring of $\mathrm{SL}(n,K)\times\mathrm{SL}(n,K)$-invariants is generated by $n+1$ algebraically independent elements over K and the action of $\mathrm{SL}(2,K)$ on that ring is identical with the one defined in the classical invariant theory of binary forms. We also reveal the ring of $\mathrm{SL}(m,K)\times\mathrm{SL}(n,K)$-invariants and $\mathrm{SL}(m,K)\times\mathrm{SL}(n,K)\times\mathrm{SL}(2,K)$-invariants completely in the case where $m\neq n$.

1 Introduction

High-dimensional array data analysis is now rapidly developing and being successfully applied in various fields. A high-dimensional array datum is called a tensor in those communities. To be precise, a d-dimensional array datum $(a_{i_1i_2\cdots i_d})_{1\leq i_j\leq m_j}$ is called a d-tensor or an $m_1\times\cdots\times m_d$-tensor.

A 2-tensor is no other than a matrix. For a matrix of indeterminates, that is, a matrix whose entries are independent indeterminates, there are many results about the action of various subgroups of the general linear group and the rings of invariants under this action.

To be precise, let $X=(X_{ij})$ be an $m\times n$ matrix of indeterminates, that is, a matrix whose entries are independent indeterminates, and G a subgroup of $\mathrm{GL}(m,K)$, where K is an infinite field. For $P\in G$, one defines

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the action of P on $K[X] = K[X_{ij} \mid 1 \le i \le m,\ 1 \le j \le n]$ by the K algebra homomorphism sending X to $P^\top X$. Let us state this another words. Let V_1 and V_2 be vector spaces over K with dimensions m and n respectively. Since G acts on V_1 linearly, one can extend it to the action of G on $\mathrm{Sym}(V_1 \otimes V_2)$, the symmetric algebra of $V_1 \otimes V_2$ over K.

In this paper, we first consider the action of the product of two special linear groups on a 3-tensor of indeterminates, that is, a 3-tensor whose entries are independent indeterminates. Let $T = (T_{ijk})_{1 \le i \le m, 1 \le j \le n, 1 \le k \le 2}$ be an $m \times n \times 2$-tensor of indeterminates. Set $X_k = (T_{ijk})_{1 \le i \le m, 1 \le j \le n}$ for $k = 1, 2$. Then $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)$ acts on the polynomial ring $K[T] = K[T_{ijk} \mid 1 \le i \le m,\ 1 \le j \le n,\ 1 \le k \le 2]$ by the K-algebra homomorphism sending X_k to $P^\top X_k Q$ for $k = 1, 2$, where $(P, Q) \in \mathrm{SL}(m,K) \times \mathrm{SL}(n,K)$.

To state it another words, let V_1, V_2 and V_3 be K-vector spaces of dimensions m, n and 2 respectively. Then the natural actions of $\mathrm{SL}(m,K)$ on V_1 and $\mathrm{SL}(n,K)$ on V_2 induces an action of $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)$ on $\mathrm{Sym}(V_1 \otimes V_2 \otimes V_3)$. And we show the following facts. (1) If $m = n \ge 2$, then $K[T]^{\mathrm{SL}(n,K) \times \mathrm{SL}(n,K)}$ is generated by $n+1$ algebraically independent elements over K. (2) If $n = m + \gcd(m,n)$, then $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)}$ is generated by one element over K. (3) If $m < n$ and $n \ne m + \gcd(m,n)$, then $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)} = K$. (See Theorems 3.8 and 3.16.)

Next we consider the action of $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K) \times \mathrm{SL}(2,K)$ on $K[T]$, in particular, the action of $\mathrm{SL}(2,K)$ on $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)}$. And we show, above all things, that in the case where $m = n$ and $K = \mathbb{C}$, this action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Sym}(V_1 \otimes V_2 \otimes V_3)^{\mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})} = \mathbb{C}[T]^{\mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})}$ coincides with the action of classical invariant theory of binary forms. See Theorem 4.8. We also make a remark on hyperdeterminant defined by Gelfand, Kapranov and Zelvinsky [GKZ1]. See Corollary 5.9. Since the theory of classical invariants dates back to nineteenth century, using the accumulated results on classical invariants of binary forms ([SF], [Hil], [Dol], [Shi], [DL], [BP1] and [BP2]) and combining the results of this paper, one obtains much information about $\mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$-invariant polynomials in $\{T_{ijk}\}_{1 \le i \le n, 1 \le j \le n, 1 \le k \le 2}$ and hyperdeterminant of format $n-1$, $n-1$, 1.

The organization of this paper is as follows. After establishing notation and recalling basic facts in Section 2, we study in Section 3 the invariants in $K[T]$ under the action of $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)$ stated above. In Section 4, we study the invariants in $K[T]$ under the action of $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K) \times \mathrm{SL}(2,K)$. And show that the following facts. (1) If $m \ne n$, then $K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K) \times \mathrm{SL}(2,K)} = K[T]^{\mathrm{SL}(m,K) \times \mathrm{SL}(n,K)}$. (2) If $m = n$ and $K = \mathbb{C}$, then the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}[T]^{\mathrm{SL}(m,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})}$ is isomorphic to the action of $\mathrm{SL}(2,\mathbb{C})$ considered in the classical invariant theory of binary forms. The assumption $K = \mathbb{C}$ is made because the classical invariant theory is studied

under this assumption.

In Section 5, we make a brief comment on the relation between the classical invariant of binary forms and the hyperdeterminant. And state a necessary condition that an $\mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$-invariant is the hyperdeterminant. See Corollary 5.9.

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2 Preliminaries

In this paper, all rings and algebras are commutative with identity element and K will always denote an infinite field. The characteristic of K is arbitrary except specified.

Notation (1) Let R be a ring and $\{a_{i_1 i_2 \cdots i_d}\}_{1 \le i_j \le m_j}$ be a family of elements in R. A d-dimensional array datum $T = (a_{i_1 i_2 \cdots i_d})_{1 \le i_j \le m_j}$ is called a d-tensor or an $m_1 \times \cdots \times m_d$-tensor and each $a_{i_1 i_2 \cdots i_d}$ is called an entry of T.

(2) Let M be an $m \times n$ matrix with entries in a ring R. We denote by $\Gamma(M)$ the set of m-minors of M (which may be empty) and by $\Gamma'(M)$ the set of n-minors of M.

(3) For a subring S of R and a tensor or a matrix T, the subring generated by S and the entries of T is denoted by $S[T]$.

(4) A matrix whose entries are independent indeterminates is called a matrix of indeterminates. A tensor of indeterminates is defined similarly.

(5) We denote by $\mathrm{lm}(f)$ the leading monomial of a polynomial f in a polynomial ring with monomial order.

(6) We denote the transposed matrix of a matrix M by $M^\top$.

(7) We denote the cardinality of a set $\mathbb{S}$ by $|\mathbb{S}|$.

(8) For a vector space V over K, we denote by $\mathrm{Sym}(V)$ the symmetric algebra of V over K.

(9) For a monomial $g = X_1^{a_1} X_2^{a_2} \cdots X_u^{a_u}$ with indeterminates $X_1, \ldots, X_u$, we set $\mathrm{supp}(g) := \{X_k \mid a_k > 0\}$.

(10) For a positive integer n, we set $[n] := \{1, 2, \ldots, n\}$.

Definition 2.1 Let m and n be positive integers. We define

$$\Gamma(m \times n) := \{[a_1, a_2, \ldots, a_m] \mid 1 \leq a_1 < a_2 < \cdots < a_m \leq n, a_i \in \mathbb{Z}\}$$

and

$$\Gamma'(m \times n) := \{\langle a_1, a_2, \ldots, a_n \rangle \mid 1 \leq a_1 < a_2 < \cdots < a_n \leq m, a_i \in \mathbb{Z}\}.$$

And we define the order on $\Gamma(m \times n)$ by

$$[a_1, \ldots, a_m] \leq [b_1, \ldots, b_m] \overset{\text{def}}{\iff} a_i \leq b_i \text{ for } 1 \leq i \leq m.$$

The order on $\Gamma'(m \times n)$ is defined similarly.

We use the terminology and basic facts on algebras with straightening law (ASL for short) freely. (See [BV], [BH] or [DEP]. Note that in [BH] and [DEP], an ASL is called an ordinal Hodge algebra.)

Definition 2.2 For an $m \times n$ matrix $M = (m_{ij})$ with entries in a ring R and $[a_1, \ldots, a_m] \in \Gamma(m \times n)$, we set

$$[a_1, \ldots, a_m]_M := \det(m_{ia_j})$$

and for a standard monomial $\mu = \prod_{k=1}^{l}[a_{k1}, \ldots, a_{km}]$ on $\Gamma(m \times n)$, we set

$$\mu_M := \prod_{k=1}^{l}[a_{k1}, \ldots, a_{km}]_M.$$

$\langle b_1, \ldots, b_n \rangle_M$ and ν_M for $\langle b_1, \ldots, b_n \rangle \in \Gamma'(m \times n)$ and a standard monomial on $\Gamma'(m \times n)$ is defined similarly.

Now let $X = (X_{ij})$ be an $m \times n$ matrix of indeterminates. First we recall the following:

Fact 2.3 *$K[\Gamma(X)]$ is an ASL on $\Gamma(m \times n)$ over K by the embedding $[a_1, \ldots, a_m] \mapsto [a_1, \ldots, a_m]_X$.*

Suppose that a diagonal monomial order is defined on $K[X]$. Then the leading monomial of $[a_1, \ldots, a_m]_X$ is $X_{1a_1}X_{2a_2} \cdots X_{ma_m}$. Moreover, the following result holds.

Lemma 2.4 (c.f. [Stu, Lemma 3.1.8]) *If μ and μ' are standard monomials on $\Gamma(m \times n)$ with $\mu \neq \mu'$, then $\mathrm{lm}(\mu_X) \neq \mathrm{lm}(\mu'_X)$. In particular, if*

$$f = \sum_k c_k \mu_k$$

is the standard representation of $f \in K[\Gamma(X)]$, there is a unique k such that $\mathrm{lm}(f) = \mathrm{lm}((\mu_k)_X)$.

We define the action of $\mathrm{SL}(m, K)$ on $K[X]$ by sending X_{ij} to the (i, j)-entry of $P^\top X$ for any i and j, where P is an element of $\mathrm{SL}(m, K)$. Note that this action coincides with the action on $\mathrm{Sym}(K^m \otimes K^n)$ induced by the natural action of $\mathrm{SL}(m, K)$ on K^m under the natural isomorphism $\mathrm{Sym}(K^m \otimes K^n) \simeq K[X]$ such that $X_{ij} \leftrightarrow \boldsymbol{e}_i^{(1)} \otimes \boldsymbol{e}_j^{(2)}$, where $\boldsymbol{e}_1^{(1)}, \ldots, \boldsymbol{e}_m^{(1)}$ and $\boldsymbol{e}_1^{(2)}, \ldots, \boldsymbol{e}_n^{(2)}$ are the canonical bases of K^m and K^n respectively.

Under this action, the following fact is known.

Theorem 2.5 (see e.g. [BV, (7.4) Proposition and (7.7) Corollary]) $K[X]^{\mathrm{SL}(m,K)} = K[\Gamma(X)]$.

3 The ring of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$-invariants

In this section, we consider the action of $\mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$ on $K[T] = K[T_{ijk} \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2]$, where $T = (T_{ijk})$ is an $m \times n \times 2$-tensor of indeterminates and study the ring of indeterminates under this action. Set $X = (T_{ij1})$ and $Y = (T_{ij2})$. Also set $G = \mathrm{SL}(m, K) \times \mathrm{SL}(n, K)$. We define the action of G by

$$(P, Q) \cdot X = P^\top X Q \quad \text{and} \quad (P, Q) \cdot Y = P^\top Y Q$$

for $(P, Q) \in G$. Note that this action coincides with the action on $\mathrm{Sym}(K^m \otimes K^n \otimes K^2)$ induced by the natural actions of $\mathrm{SL}(m, K)$ on K^m and $\mathrm{SL}(n, K)$ on K^n under the natural isomorphism $\mathrm{Sym}(K^m \otimes K^n \otimes K^2) \simeq K[T]$ with $T_{ijk} \leftrightarrow \boldsymbol{e}_i^{(1)} \otimes \boldsymbol{e}_j^{(2)} \otimes \boldsymbol{e}_k^{(3)}$, where $\boldsymbol{e}_1^{(1)}, \ldots, \boldsymbol{e}_m^{(1)}$; $\boldsymbol{e}_1^{(2)}, \ldots, \boldsymbol{e}_n^{(2)}$ and $\boldsymbol{e}_1^{(3)}, \boldsymbol{e}_2^{(3)}$ are the canonical bases of K^m, K^n and K^2 respectively.

By Theorem 2.5 and symmetry, it is clear that the following fact holds.

Lemma 3.1 $K[T]^G = K[\Gamma(X, Y)] \cap K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$.

By symmetry, we may assume that $m \leq n$. If $n > 2m$, then $K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}] = K$, so in the following, we also assume that $n \leq 2m$.

We introduce the degree lexicographic monomial order on $K[T]$ with $T_{111} > T_{211} > T_{311} > \cdots > T_{m11} > T_{121} > \cdots > T_{m21} > T_{131} > \cdots > T_{mn1} > T_{112} > T_{212} > \cdots > T_{m12} > T_{122} > \cdots > T_{mn2}$. And we set $Z = (X, Y) = (Z_{ij})$ and $W = \begin{pmatrix} X \\ Y \end{pmatrix} = (W_{ij})$. Note the monomial order we defined is diagonal both for Z and W. So we see the following:

Lemma 3.2 *For an element* $[a_1, \ldots, a_m] \in \Gamma(m \times 2n)$, $\mathrm{lm}([a_1, \ldots, a_m]_Z) = Z_{1a_1} Z_{2a_2} \cdots Z_{ma_m}$ *and for* $\langle b_1, \ldots, b_n \rangle \in \Gamma'(2m \times n)$, $\mathrm{lm}(\langle b_1, \ldots, b_n \rangle_W) = W_{b_1 1} W_{b_2 2} \cdots W_{b_n n}$.

Let f be a non-zero element of $K[\Gamma(X, Y)] \cap K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$. Since $f \in K[\Gamma(X, Y)]$, by Lemmas 2.4 and 3.2, we see that $\mathrm{supp}(\mathrm{lm}(f)) \cap \{Z_{ij} \mid j < i$ or $2n - j < m - i\} = \emptyset$. In other words,

$$\mathrm{supp}(\mathrm{lm}(f)) \cap (\{T_{ij1} \mid j < i\} \cup \{T_{ij2} \mid n - j < m - i\}) = \emptyset.$$

By symmetry, we also see that

$$\mathrm{supp}(\mathrm{lm}(f)) \cap (\{T_{ij1} \mid i < j\} \cup \{T_{ij2} \mid m - i < n - j\}) = \emptyset.$$

Therefore, we see the following fact:

Lemma 3.3 *Let f be a non-zero element of* $K[\Gamma(X, Y)] \cap K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$. *Then*

$$\mathrm{supp}(\mathrm{lm}(f)) \subset \{T_{111}, T_{221}, \ldots, T_{mm1}, T_{1,n-m+1,2}, T_{2,n-m+2,2}, \ldots, T_{mn2}\}.$$

In other words, $\mathrm{supp}(\mathrm{lm}(f)) \subset \{Z_{ij} \mid i = j$ *or* $m - i = 2n - j\}$ *and* $\mathrm{supp}(\mathrm{lm}(f)) \subset \{W_{ij} \mid i = j \leq m$ *or* $2m - i = n - j < m\}$.

By Lemmas 2.4, 3.2 and 3.3 we see the following:

Lemma 3.4 *Let f be a non-zero element of* $K[\Gamma(X, Y)] \cap K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$. *Also let*

$$f = \sum_k c_k \mu_k \quad \textit{and} \quad f = \sum_l d_l \nu_l$$

be the standard representations of f with respect to the ASL structure of $K[\Gamma(X, Y)]$ *on* $\Gamma(m \times 2n)$ *and* $K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$ *on* $\Gamma'(2m \times n)$ *respectively. Suppose*

that $\mu := \mu_k$ *is the standard monomial on* $\Gamma(m \times 2n)$ *such that* $\mathrm{lm}(f) = \mathrm{lm}(\mu_Z)$ *and* $\nu := \nu_l$ *is the standard monomial on* $\Gamma'(2m \times n)$ *such that* $\mathrm{lm}(f) = \mathrm{lm}(\nu_W)$*, then* μ *and* ν *is the following form.*

$$\mu = \prod_{t=1}^{u} [1, 2, \ldots, i_t, 2n - m + i_t + 1, 2n - m + i_t + 2, \ldots, 2n]$$

and

$$\nu = \prod_{s=1}^{v} \langle 1, 2, \ldots, j_s, 2m - n + j_s + 1, 2m - n + j_s + 2, \ldots, 2m \rangle,$$

where $i_1, \ldots, i_u$ *and* $j_1, \ldots, j_v$ *are integers with* $m \geq i_1 \geq \cdots \geq i_u \geq 0$ *and* $m \geq j_1 \geq \cdots \geq j_v \leq n - m$*. And*

$$\begin{aligned} \mathrm{lm}(f) &= \prod_{t=1}^{u} T_{111} \cdots T_{i_t i_t 1} T_{i_t+1, n-m+i_t+1, 2} \cdots T_{mn2} \\ &= \prod_{s=1}^{v} T_{111} \cdots T_{j_s j_s 1} T_{m-n+j_s+1, j_s+1, 2} \cdots T_{mn2}. \end{aligned}$$

Now we study $K[T]^G$. First we consider the case where $m = n$.

Definition 3.5 Set

$$\det(X + Y) = f_{n,0} + f_{n-1,1} + \cdots + f_{0,n},$$

where $f_{k,n-k}$ is the sum of monomials whose degree with respect to T_{ij1} and T_{ij2} is k and $n - k$ respectively.

Remark 3.6 Set $X = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ and $Y = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n]$ and for $\mathbb{S} \subset [n]$, set

$$\boldsymbol{z}_j^{\mathbb{S}} := \begin{cases} \boldsymbol{x}_j & (j \in \mathbb{S}) \\ \boldsymbol{y}_j & (j \notin \mathbb{S}). \end{cases}$$

Then

$$f_{k,n-k} = \sum_{\mathbb{S} \subset [n], |\mathbb{S}| = k} \det[\boldsymbol{z}_1^{\mathbb{S}}, \ldots, \boldsymbol{z}_n^{\mathbb{S}}].$$

In particular, $\mathrm{lm}(f_{k,n-k}) = T_{111} \cdots T_{kk1} T_{k+1,k+1,2} \cdots T_{nn2}$, and therefore leading monomials of $f_{n,0}, f_{n-1,1}, \ldots, f_{0,n}$ are algebraically independent over K. It follows that $f_{n,0}, f_{n-1,1}, \ldots, f_{0,n}$ are algebraically independent over K.

Next we state the following:

Lemma 3.7 $K[\det(cX + dY) \mid c, d \in K] = K[f_{n,0}, f_{n-1,1}, \dots, f_{0,n}]$.

Proof Since $\det(cX+dY) = \sum_{k=0}^{n} c^{n-k} d^k f_{n-k,k}$, it is clear that $K[\det(cX+dY) \mid c, d \in K] \subset K[f_{n,0}, f_{n-1,1}, \dots, f_{0,n}]$. On the other hand, since $\det(cX+Y) = \sum_{k=0}^{m} c^{n-k} f_{n-k,k}$ and K is an infinite field, by the argument using Vandermonde determinant, we see that $f_{n-k,k} \in K[\det(cX+Y) \mid c \in K]$ for any k. ∎

Now we state the following:

Theorem 3.8 *Suppose $m = n$. Then $f_{n,0}, f_{n-1,1}, \dots, f_{0,n}$ is a sagbi basis of $K[T]^G$. In particular, $K[T]^G = K[f_{n,0}, f_{n-1,1}, \dots, f_{0,n}]$ and $K[T]^G$ is isomorphic to the polynomial ring with $n+1$ variables over K.*

Proof Clearly, for any c and $d \in K$, $\det(cX+dY)$ is invariant under the action of G. Therefore, $K[T]^G \supset K[\det(cX+dY) \mid c, d \in K] = K[f_{n,0}, f_{n-1,1}, \dots, f_{0,n}]$ by Lemma 3.7. On the other hand, for any $f \in K[\Gamma(X,Y)] \cap K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$ with $f \neq 0$, we see that $\mathrm{lm}(f)$ is contained in $K[\mathrm{lm}(f_{n,0}), \dots, \mathrm{lm}(f_{0,n})]$ by Lemma 3.4 and Remark 3.6. So the second sentence of the theorem follows by Lemma 3.1. The third sentence follows from the basic property of sagbi basis and Remark 3.6. ∎

Next we consider the case where $m < n \leq 2m$. Let f a non-zero element of $K[\Gamma(X,Y)] \cap K[\Gamma' \begin{pmatrix} X \\ Y \end{pmatrix}]$. We use the notation of Lemma 3.4.

Definition 3.9 For an integer i with $0 \leq i \leq m$, we set

$$[[i, m-i]] := [1, 2, \dots, i, 2n-m+i+1, 2n-m+i+2, \dots, 2n] \in \Gamma(m \times 2n)$$

and for an integer j with $n-m \leq j \leq m$, we set

$$\langle\langle j, n-j\rangle\rangle := \langle 1, 2, \dots, j, 2m-n+j+1, 2m-n+j+2, \dots, 2m\rangle \in \Gamma'(2m \times n).$$

Remark 3.10 By Lemma 3.4, we see that $\mu = \prod_{t=1}^{u} [[i_t, m-i_t]]$ and $\nu = \prod_{s=1}^{v} \langle\langle j_s, n-j_s\rangle\rangle$.

Definition 3.11 Set $l = n - m$. Also set

$$\mu = \prod_{i=0}^{m} [[i, m-i]]^{c_i} \quad \text{and} \quad \nu = \prod_{j=l}^{m} \langle\langle j, n-j\rangle\rangle^{d_j},$$

that is, $c_i = |\{t \mid i_t = i\}|$ and $d_j = |\{s \mid j_s = j\}|$. We define

$$L(i, m-i) := c_i \quad \text{for } i = 0,\, 1,\, \ldots,\, m$$

and

$$L\begin{pmatrix} j \\ n-j \end{pmatrix} := d_j \quad \text{for } j = l,\, l+1,\, \ldots,\, m.$$

By comparing the exponent of T_{mm1} of $\mathrm{lm}(f) = \mathrm{lm}(\mu_Z) = \mathrm{lm}(\nu_W)$, we see that

$$L(m, 0) = L\begin{pmatrix} m \\ l \end{pmatrix}.$$

Similarly, by comparing the exponent of $T_{m-1,m-1,1}$, we see that

$$L(m-1, 1) + L(m, 0) = L\begin{pmatrix} m-1 \\ l+1 \end{pmatrix} + L\begin{pmatrix} m \\ l \end{pmatrix}.$$

Therefore,

$$L(m-1, 1) = L\begin{pmatrix} m-1 \\ l+1 \end{pmatrix}.$$

By continuing the same argument, we see that

$$L(m-k, k) = L\begin{pmatrix} m-k \\ l+k \end{pmatrix} \quad \text{for } k = 0,\, 1,\, \ldots,\, m-l. \tag{3.1}$$

By symmetry, we also see that

$$L(k, m-k) = L\begin{pmatrix} l+k \\ m-k \end{pmatrix} \quad \text{for } k = 0,\, 1,\, \ldots,\, m-l. \tag{3.2}$$

Next, by comparing the exponent of $T_{l-1,l-1,1}$ of $\mathrm{lm}(f) = \mathrm{lm}(\mu_Z) = \mathrm{lm}(\nu_W)$, we obtain the following equation.

$$\begin{aligned} & L(l-1, m-l+1) + L(l, m-l) + \cdots + L(m, 0) \\ = \;& L\begin{pmatrix} l \\ m-l \end{pmatrix} + L\begin{pmatrix} l+1 \\ m-l-1 \end{pmatrix} + \cdots + L\begin{pmatrix} m \\ l \end{pmatrix}. \end{aligned}$$

Therefore we see, by equation (3.1), that

$$L(l-1, m-l+1) = 0.$$

Similarly, by comparing the exponent of $T_{l-2,l-2,1}$, we see that

$$\begin{aligned} & L(l-2, m-l+2) + L(l-1, m-l+1) + \cdots + L(m, 0) \\ = \;& L\begin{pmatrix} l \\ m-l \end{pmatrix} + L\begin{pmatrix} l+1 \\ m-l-1 \end{pmatrix} + \cdots + L\begin{pmatrix} m \\ l \end{pmatrix} \end{aligned}$$

and from this, we see that

$$L(l-2, m-l+2) = 0.$$

By continuing this argument, we see that

$$L(1, m-1) = L(2, m-2) = \cdots = L(l-1, m-l+1) = 0. \tag{3.3}$$

Let r be an integer with $0 < r < l$. By equations (3.3), (3.2) and (3.1), we see that

$$\begin{aligned} 0 = L(r, m-r) &= L\begin{pmatrix} r+l \\ m-r \end{pmatrix} = L(r+l, m-r-l) \\ &= L\begin{pmatrix} r+2l \\ m-r-l \end{pmatrix} = L(r+2l, m-r-2l) \\ &= \cdots \\ &= L\begin{pmatrix} r+ql \\ m-r-(q-1)l \end{pmatrix} = L(r+ql, m-r-ql), \end{aligned}$$

where $q = \lfloor (m-r)/l \rfloor$. By this equation and symmetry, we see the following:

Lemma 3.12 *Let k be an integer with $1 \leq k \leq m$. If $l \nmid k$, then $L(k, m-k) = L(m-k, k) = 0$.*

If $l \nmid m$, then by Lemma 3.12, we see that $L(0, m) = 0$. Therefore,

$$\begin{aligned} 0 = L(0, m) &= L\begin{pmatrix} l \\ m \end{pmatrix} = L(l, m-l) \\ &= L\begin{pmatrix} 2l \\ m-l \end{pmatrix} = L(2l, m-2l) \\ &= \cdots \\ &= L\begin{pmatrix} ql \\ m-(q-1)l \end{pmatrix} = L(ql, m-ql), \end{aligned}$$

where $q = \lfloor m/l \rfloor$. By this equation and Lemma 3.12, we see the following:

Lemma 3.13 *If $l \nmid m$, then*

$$L(k, m-k) = 0 \quad \text{for } k = 0,\ 1,\ \ldots,\ m.$$

In particular, $f \in K$.

Set $d = \gcd(m, n)$. Note that $l|m$ if and only if $d = l$. Now consider the case where $l|m$. Set $c = L(0, m)$. Then by equations (3.2) and (3.1), we see that

$$\begin{aligned} c = L(0,m) &= L\begin{pmatrix} d \\ m \end{pmatrix} = L(d, m-d) \\ &= L\begin{pmatrix} 2d \\ m-d \end{pmatrix} = L(2d, m-2d) \\ &= \cdots \\ &= L\begin{pmatrix} m \\ d \end{pmatrix} = L(m, 0), \end{aligned}$$

Therefore, by Lemmas 3.2 and 3.4, we see the following:

Lemma 3.14 *Suppose $l = d$ and set $s = m/d$. Then*

$$\begin{aligned} \mathrm{lm}(f) &= (T_{111}T_{221}\cdots T_{dd1})^{sc} \\ &\quad \times (T_{d+1,d+1,1}T_{d+2,d+2,1}\cdots T_{2d,2d,1})^{(s-1)c} \\ &\quad \times \cdots \\ &\quad \times (T_{m-d+1,m-d+1,1}T_{m-d+2,m-d+2,1}\cdots T_{mm1})^{c} \\ &\quad \times (T_{1,d+1,2}T_{2,d+2,2}\cdots T_{d,2d,2})^{c} \\ &\quad \times (T_{d+1,2d+1,2}T_{d+2,2d+2,2}\cdots T_{2d,3d,2})^{2c} \\ &\quad \times \cdots \\ &\quad \times (T_{m-d+1,n-d+1,2}T_{m-d+2,n-d+2,2}\cdots T_{mn2})^{sc}. \end{aligned}$$

Consider the following $(mn/d) \times (mn/d)$ determinant, where m/d blocks of n columns are aligned horizontally and n/d blocks of m rows are aligned vertically.

$$\begin{vmatrix} X & & & & \\ Y & X & & & \\ & Y & X & & \\ & & Y & \ddots & \\ & & & \ddots & X \\ & & & & Y \end{vmatrix} \tag{3.4}$$

Note that the Laplace expansion by n/d blocks of m rows of this determinant shows that this determinant is $\mathrm{SL}(m, K)$-invariant and the Laplace expansion by m/d blocks of n columns of the determinant shows that this determinant is $\mathrm{SL}(n, K)$-invariant. That is, the determinant (3.4) is G-invariant. We also

see that the leading monomial of the determinant (3.4) is

$$
\begin{array}{ll}
 & (T_{111}T_{221}\cdots T_{dd1})^s \\
\times & (T_{d+1,d+1,1}T_{d+2,d+2,1}\cdots T_{2d,2d,1})^{(s-1)} \\
\times & \cdots \\
\times & (T_{m-d+1,m-d+1,1}T_{m-d+2,m-d+2,1}\cdots T_{mm1}) \\
\times & (T_{1,d+1,2}T_{2,d+2,2}\cdots T_{d,2d,2}) \\
\times & (T_{d+1,2d+1,2}T_{d+2,2d+2,2}\cdots T_{2d,3d,2})^2 \\
\times & \cdots \\
\times & (T_{m-d+1,n-d+1,2}T_{m-d+2,n-d+2,2}\cdots T_{mn2})^s
\end{array}
$$

by examining the Laplace expansion by n/d blocks of m rows.

Therefore, by Lemma 3.14, we see the following:

Lemma 3.15 $\mathrm{lm}(f)$ *is a power of the leading monomial of the determinant (3.4).*

By Lemmas 3.13 and 3.15, we see the following:

Theorem 3.16 *Suppose $m < n$ and set $d = \gcd(m,n)$.*

(1) If $n = m + d$, then the determinant (3.4) is a sagbi basis of $K[T]^G$. In particular, $K[T]^G$ is generated by the determinant (3.4).

(2) If $n \neq m + d$, then the ring of G-invariants in $K[T]$ is K.

4 The action of $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K) \times \mathrm{SL}(2,K)$ and invariants of binary forms

Next we consider the action of $\mathrm{SL}(m,K) \times \mathrm{SL}(n,K) \times \mathrm{SL}(2,K)$ on $K[T]$. We define the action of $G = \mathrm{SL}(m,K) \times \mathrm{SL}(n,K)$ on $K[T]$ by the same way as in the previous section and define the action of $\mathrm{SL}(2,K)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,K)$.

Remark 4.1 The action of $\mathrm{SL}(m,K)\times\mathrm{SL}(n,K)\times\mathrm{SL}(2,K)$ on $K[T]$ defined above and in the previous section coincides with the action on $\mathrm{Sym}(K^m \otimes K^n \otimes K^2)$ induced by the natural actions of $\mathrm{SL}(m,K)$ on K^m, $\mathrm{SL}(n,K)$ on K^n and $\mathrm{SL}(2,K)$ on K^2 under the natural isomorphism $\mathrm{Sym}(K^m \otimes K^n \otimes K^2) \simeq K[T]$ with $T_{ijk} \leftrightarrow \boldsymbol{e}_i^{(1)} \otimes \boldsymbol{e}_j^{(2)} \otimes \boldsymbol{e}_k^{(3)}$, where $\boldsymbol{e}_1^{(1)}, \ldots, \boldsymbol{e}_m^{(1)}$; $\boldsymbol{e}_1^{(2)}, \ldots, \boldsymbol{e}_n^{(2)}$ and $\boldsymbol{e}_1^{(3)}, \boldsymbol{e}_2^{(3)}$ are the canonical bases of K^m, K^n and K^2 respectively.

We first consider the case where $m < n$. First we note the following:

Lemma 4.2 *Let s be a positive integer. And let A be the $(s+1)m \times sn$ matix of the following form.*

$$\begin{bmatrix} X & & & & \\ Y & X & & & \\ & Y & X & & \\ & & Y & \ddots & \\ & & & \ddots & X \\ & & & & Y \end{bmatrix}.$$

Then for any $c \in K$, there is a sequence of blockwise row and column operations such that adding a scalar multiple of j-th row block to i-th row block with $i < j$ and adding a scalar multiple of i-th column block to j-th column block with $i < j$ and transform A into

$$\begin{bmatrix} X+cY & & & & \\ Y & X+cY & & & \\ & Y & X+cY & & \\ & & Y & \ddots & \\ & & & \ddots & X+cY \\ & & & & Y \end{bmatrix}.$$

Proof We prove by induction on s. The case where $s = 1$ is clear. Assume $s > 1$ and let B be the $sm \times (s-1)n$ matrix of the following form

$$\begin{bmatrix} X & & & & \\ Y & X & & & \\ & Y & X & & \\ & & Y & \ddots & \\ & & & \ddots & X \\ & & & & Y \end{bmatrix}.$$

By the induction hypothesis, there is a sequence of row and column operations of the type stated in the lemma which transforms B to

$$\begin{bmatrix} X+cY & & & & \\ Y & X+cY & & & \\ & Y & X+cY & & \\ & & Y & \ddots & \\ & & & \ddots & X+cY \\ & & & & Y \end{bmatrix}.$$

By applying this sequence of row and column operations to the upper left corner of A, one gets

$$\begin{bmatrix} X+cY & & & & & a_1X \\ Y & X+cY & & & & a_2X \\ & Y & X+cY & & & a_3X \\ & & Y & \ddots & & \vdots \\ & & & \ddots & X+cY & a_{s-2}X \\ & & & & Y & X \\ & & & & & Y \end{bmatrix}. \tag{4.1}$$

By adding $-a_i$ times of the i-th column block to the s-th column block, one can transform the matrix (4.1) to the following form.

$$\begin{bmatrix} X+cY & & & & & b_1Y \\ Y & X+cY & & & & b_2Y \\ & Y & X+cY & & & b_3Y \\ & & Y & \ddots & & \vdots \\ & & & \ddots & X+cY & b_{s-2}Y \\ & & & & Y & X+b_{s-1}Y \\ & & & & & Y \end{bmatrix}$$

Finally, by adding $-b_i$ times of $(s+1)$-th row block to i-th row block for $i = 1, \ldots, s-2$ and $-b_{s-1}+c$ times of $(s+1)$-th row block to s-th row block, one gets a desired form. ∎

As a corollary, we see the following fact.

Corollary 4.3 *Set $d = \gcd(m, n)$ and assume that $n = m + d$. Let A be the $(mn/d) \times (mn/d)$ matrix of the following form.*

$$\begin{bmatrix} X & & & & & \\ Y & X & & & & \\ & Y & X & & & \\ & & Y & \ddots & & \\ & & & \ddots & X & \\ & & & & Y & \end{bmatrix}$$

Then the determinant of the following $(mn/d) \times (mn/d)$ matrix

$$\begin{bmatrix} X + cY & & & & & \\ Y & X + cY & & & & \\ & Y & X + cY & & & \\ & & Y & \ddots & & \\ & & & \ddots & X + cY & \\ & & & & Y & \end{bmatrix}$$

is $\det A$.

Since $\mathrm{SL}(2, K)$ is generated by $\{\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in K\} \cup \{\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in K\}$, by Corollary 4.3 and symmetry, we see the following:

Proposition 4.4 *Set $d = \gcd(m, n)$ and assume that $n = m + d$. Then the determinant (3.4) is invariant under the action of* $\mathrm{SL}(2, K)$.

Therefore, we see the following:

Theorem 4.5 *If $m < n$, then $K[T]^{G \times \mathrm{SL}(2,K)} = K[T]^G$.*

Next we consider the case where $m = n$.

Professor Mitsuyasu Hashimoto kindly informed the author that this action of $\mathrm{SL}(2, K)$ on $K[f_{n,0}, f_{n-1,1}, \ldots, f_{0,n}]$ is identical with the one of the classical invariant theory of binary forms. Here we recall the classical invariant of binary forms (see [Stu, Section 3.6] and [Muk, Section 1.3]).

From now on we assume that $K = \mathbb{C}$, the complex number field. Let x and y be indeterminates. For $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ we define $\bar{x}$ and $\bar{y}$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

Now let ξ_0, ξ_1, ..., ξ_n be $n+1$ new indeterminates and let f be the following form of x and y of degree n.

$$f = \sum_{k=0}^{n} \binom{n}{k} \xi_k x^k y^{n-k}$$

One can rewrite f by using $\bar{x}$ and $\bar{y}$ by substituting $x = \alpha\bar{x} + \beta\bar{y}$ and $y = \gamma\bar{x} + \delta\bar{y}$.

$$f = \sum_{k=0}^{n} \binom{n}{k} \xi_k (\alpha\bar{x} + \beta\bar{y})^k (\gamma\bar{x} + \delta\bar{y})^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \bar{\xi}_k \bar{x}^k \bar{y}^{n-k}$$

Definition 4.6 The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}[\xi_0, \xi_1, \ldots, \xi_n, x, y]$ is defined so that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ maps x to $\bar{x}$, y to $\bar{y}$ and ξ_k to $\bar{\xi}_k$ for $k = 0, 1, \ldots, n$. An element of $\mathbb{C}[\xi_0, \ldots, \xi_n, x, y]^{\mathrm{SL}(2,\mathbb{C})}$ is called a covariant and an element of $\mathbb{C}[\xi_0, \ldots, \xi_n]^{\mathrm{SL}(2,\mathbb{C})}$ is called an invariant.

Recall that under the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}[T]$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ maps X to $\alpha X + \gamma Y$ and Y to $\beta X + \delta Y$. Set $\bar{X} = \alpha X + \gamma Y$ and $\bar{Y} = \beta X + \delta Y$. Then since $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$, we see that

$$\bar{x}\bar{X} + \bar{y}\bar{Y} = xX + yY.$$

Set $\bar{X} = [\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_n]$ and $\bar{Y} = [\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_n]$ and for $\mathbb{S} \subset [n]$, set

$$\bar{\boldsymbol{z}}_j^{\mathbb{S}} := \begin{cases} \bar{\boldsymbol{x}}_j & (j \in \mathbb{S}) \\ \bar{\boldsymbol{y}}_j & (j \notin \mathbb{S}). \end{cases}$$

Also set

$$\bar{f}_{k,n-k} = \sum_{\mathbb{S} \subset [n], |\mathbb{S}| = k} \det[\bar{\boldsymbol{z}}_1^{\mathbb{S}}, \ldots, \bar{\boldsymbol{z}}_n^{\mathbb{S}}].$$

Since by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ are mapped to $\bar{\boldsymbol{x}}_k$ and $\bar{\boldsymbol{y}}_k$ respectively for $k = 0, \ldots, n$, we see, by Remark 3.6, the following:

Lemma 4.7 *$f_{k,n-k}$ is mapped to $\bar{f}_{k,n-k}$ by the action of* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$.

On the other hand, by expanding $\det(xX + yY)$ and $\det(\bar{x}\bar{X} + \bar{y}\bar{Y})$, one obtains

$$\sum_{k=0}^{n} f_{k,n-k} x^k y^{n-k} = \det(xX + yY) = \det(\bar{x}\bar{X} + \bar{y}\bar{Y}) = \sum_{k=0}^{n} \bar{f}_{k,n-k} \bar{x}^k \bar{y}^{n-k}.$$

Therefore, we see the following:

Theorem 4.8 *$\mathbb{C}[f_{n,0}, f_{n-1,1}, \ldots, f_{0,n}]$ is isomorphic to $\mathbb{C}[\xi_0, \xi_1, \ldots, \xi_n]$ as $\mathrm{SL}(2,\mathbb{C})$-modules by the map $f_{k,n-k} \mapsto \binom{n}{k}\xi_k$ for $k = 0, 1, \ldots, n$.*

The theory of classical invariants of binary forms dates back to nineteenth century. And is still a theme of research in progress. See [SF], [Hil], [Dol], [Shi], [DL], [BP1] and [BP2]. And by using these results, we obtain the information of the structure of $\mathbb{C}[T]^{G\times\mathrm{SL}(2,\mathbb{C})}$.

Example 4.9 (1) The ring of invariants of a binary quadric is generated by $\xi_1^2 - \xi_0\xi_2$ over $\mathbb{C}$. Therefore, if $m = n = 2$, then $\mathbb{C}[T]^{G\times\mathrm{SL}(2,\mathbb{C})}$ is generated by $f_{11}^2 - 4f_{02}f_{20}$ over $\mathbb{C}$.

(2) The ring of invariants of a binary cubic is generated by $3\xi_1^2\xi_2^2 - 4\xi_0\xi_2^3 - 4\xi_1^3\xi_3 + 6\xi_0\xi_1\xi_2\xi_3 - \xi_0^2\xi_3^2$ over $\mathbb{C}$. Therefore, if $m = n = 3$, then $\mathbb{C}[T]^{G\times\mathrm{SL}(2,\mathbb{C})}$ is generated by $f_{12}^2f_{21}^2 - 4f_{03}f_{12}^3 - 4f_{12}^3f_{30} + 18f_{03}f_{12}f_{21}f_{30} - 27f_{03}^2f_{30}^2$ over $\mathbb{C}$.

(3) The ring of invariants of a binary quartic is generated by $3\xi_2^2 - 4\xi_1\xi_3 + \xi_0\xi_4$ and $\xi_2^3 + \xi_0\xi_3^2 + \xi_1^2\xi_4 - 2\xi_1\xi_2\xi_3 - \xi_0\xi_2\xi_4$ over $\mathbb{C}$. Therefore, if $m = n = 4$, then $\mathbb{C}[T]^{G\times\mathrm{SL}(2,\mathbb{C})}$ is generated by $f_{22}^2 - 3f_{13}f_{31} + 12f_{04}f_{40}$ and $2f_{22}^3 + 27f_{04}f_{31}^2 + 27f_{13}^2f_{40} - 9f_{13}f_{22}f_{31} - 72f_{04}f_{22}f_{40}$ over $\mathbb{C}$.

5 Hyperdeterminant

In this section, we make a brief remark on the relation to hyperdeterminants defined by Gelfand, Kapranov and Zelvinsky [GKZ1] (see also [GKZ2]). First we recall the definition of the hyperdeterminant.

Definition 5.1 ([GKZ1]) Let $l_1, l_2, \ldots, l_d$ be positive integers and let V be the image of the Segre embedding $\mathbb{P}_{\mathbb{C}}^{l_1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{l_d} \to \mathbb{P}_{\mathbb{C}}^{(l_1+1)\cdots(l_d+1)-1}$. If the projective dual variety $V^\vee$ is a hypersurface, then the defining polynomial of $V^\vee$ is called the hyperdeterminant of format $l_1, \ldots, l_d$.

Remark 5.2 $V^\vee$ is a variety in the dual space $(\mathbb{P}_{\mathbb{C}}^{(l_1+1)\cdots(l_d+1)-1})^*$. Therefore, if $V_1, \ldots, V_d$ are $\mathbb{C}$-vector spaces of dimension $l_1+1, \ldots, l_d+1$ respectively, then $V^\vee$ is a subvariety of $\mathbb{P}(V_1^* \otimes \cdots \otimes V_d^*)$. In particular, if $V^\vee$ is a hypersurface, then the defining polynomial of $V^\vee$ is an element of $\mathrm{Sym}(V_1 \otimes \cdots \otimes V_d)$. Therefore, in our terminology, a hyperdeterminant of format $l_1, \ldots, l_d$ is a polynomial of the entries of $(l_1+1) \times \cdots \times (l_d+1)$-tensor of indeterminates.

Now we recall some results of [GKZ1].

Theorem 5.3 ([GKZ1, Theorem 1.3]) *The hyperdeterminant of format $l_1, \ldots, l_d$ exists if and only if*

$$l_k \leq \sum_{j \neq k} l_j$$

for any $k = 1, \ldots, d$.

Proposition 5.4 (c.f. [GKZ1, Proposition 1.4]) *The hyperdeterminant of format $l_1, \ldots, l_d$ is invariant under the action of* $\mathrm{SL}(l_1+1, \mathbb{C}) \times \cdots \times \mathrm{SL}(l_d+1, \mathbb{C})$.

In order to state the next proposition, we make the following:

Definition 5.5 Let T be an $(l_1+1) \times \cdots \times (l_d+1)$-tensor of indeterminates. For a monomial g of $K[T]$, we define the index support $\mathrm{isupp}(g)$ of g as follows:

$$\mathrm{isupp}(g) := \{(i_1, \ldots, i_d) \mid T_{i_1,\ldots,i_d} \in \mathrm{supp}(g)\}.$$

Proposition 5.6 (c.f. [GKZ1, Proposition 1.8]) *Let P be a homogeneous element of $K[T]$ which is invariant under the action of* $\mathrm{SL}(l_1+1, \mathbb{C}) \times \cdots \times \mathrm{SL}(l_d+1, \mathbb{C})$. *Then P is divisible by the hyperdeterminant of format $l_1, \ldots, l_d$ if and only if for any monomial g of P and any $(j_1, \ldots, j_d) \in [l_1+1] \times \cdots \times [l_d+1]$, there exists $(i_1, \ldots, i_d) \in \mathrm{isupp}(g)$ such that $|\{k \mid i_k \neq j_k\}| \leq 1$.*

From now on, we consider the case where $d = 3$, $l_1 = l_2$ and $l_3 = 1$ and we set $n = l_1 + 1$. We recall the following result which is a special case of [GKZ1, Corollary 3.8].

Proposition 5.7 *The degree of the hyperdeterminant of format $n-1$, $n-1$, 1 is $2n(n-1)$.*

Since the hyperdeterminant of format $n-1$, $n-1$, 1 is invariant under the action of $\mathrm{SL}(n,\mathbb{C})\times\mathrm{SL}(n,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$, it can be expressed as a polynomial of $f_{0,n}$, $f_{1,n-1}$, $\ldots$, $f_{n,0}$ by Theorem 3.8.

Let U_0, U_1, $\ldots$, U_n be indeterminates. We introduce the degree reverse lexicographic order with $U_n > U_{n-1} > \cdots > U_0$ on $\mathbb{C}[U_0,\ldots,U_n]$. Note that for $g \in \mathbb{C}[U_0,\ldots,U_n]$ with $g \neq 0$, $g(f_{0,n}, f_{1,n-1}, \ldots, f_{n,0}) \neq 0$, since $f_{0,n}$, $\ldots$, $f_{n,0}$ are algebraically independent over $\mathbb{C}$.

Lemma 5.8 *Let a_0, a_1, $\ldots$, a_n and b_0, b_1, $\ldots$, b_n be non-negative integers. Then*

$$\mathrm{lm}(f_{0,n})^{a_0}\cdots\mathrm{lm}(f_{n,0})^{a_n} < \mathrm{lm}(f_{0,n})^{b_0}\cdots\mathrm{lm}(f_{n,0})^{b_n}$$

if and only if

$$U_0^{a_0}\cdots U_n^{a_n} < U_0^{b_0}\cdots U_n^{b_n}.$$

In particular, for any $g \in \mathbb{C}[U_0, \ldots, U_n]$ with $g \neq 0$,

$$\mathrm{lm}(g(f_{0,n},\ldots,f_{n,0})) = \mathrm{lm}(g)(\mathrm{lm}(f_{0,n}),\ldots,\mathrm{lm}(f_{n,0})).$$

Proof Since $\mathrm{lm}(f_{k,n-k}) = T_{111}\cdots T_{kk1}T_{k+1,k+1,2}\cdots T_{nn2}$, the exponent of T_{kk1} of $\mathrm{lm}(f_{0,n})^{c_0}\cdots\mathrm{lm}(f_{n,0})^{c_n} = c_k + c_{k+1} + \cdots + c_n$ for any non-negative integers c_0, c_1, $\ldots$, c_n. Therefore,

$$\mathrm{lm}(f_{0,n})^{a_0}\cdots\mathrm{lm}(f_{n,0})^{a_n} < \mathrm{lm}(f_{0,n})^{b_0}\cdots\mathrm{lm}(f_{n,0})^{b_n}$$

if and only if there exists h with $0 \leq h \leq n$ such that $\sum_{k=h}^n a_k < \sum_{k=h}^n b_k$ and $\sum_{k=h'}^n a_k = \sum_{k=h'}^n b_k$ for any h' with $0 \leq h' < h$. That is,

$$U_0^{a_0}\cdots U_n^{a_n} < U_0^{b_0}\cdots U_n^{b_n}.$$

The last sentence of the lemma follows from the first part of the lemma. ∎

Now we state a corollary of Proposition 5.6.

Corollary 5.9 *Let $\Phi(U_0, U_1, \ldots, U_n)$ be the element of $\mathbb{C}[U_0, \ldots, U_n]$ such that $\Phi(f_{0,n}, f_{1,n-1}, \ldots, f_{n,0})$ is the hyperdeterminant of format $n-1$, $n-1$, 1. Then* $\mathrm{supp}(\mathrm{lm}(\Phi)) \cap \{U_0, U_1\} \neq \emptyset$.

Proof Assume the contrary. Then since $\mathrm{lm}(f_{k,n-k}) = T_{111}\cdots T_{kk1}T_{k+1,k+1,2}\cdots T_{nn2}$ for $k = 2, 3, \ldots, n$, we see, by Lemma 5.8, that

$$\begin{aligned}
&\mathrm{supp}(\mathrm{lm}(\Phi(f_{0,n},\ldots,f_{n,0}))) \\
=\ &\mathrm{supp}(\mathrm{lm}(\Phi)(\mathrm{lm}(f_{0,n}),\ldots,\mathrm{lm}(f_{n,0}))) \\
\subset\ &\{T_{111}, T_{221}, \ldots, T_{nn1}, T_{332}, T_{442}, \ldots, T_{nn2}\}.
\end{aligned}$$

Therefore, we see that the pair $\mathrm{lm}(\Phi(f_{0,n},\ldots,f_{n,0}))$ and $(1,2,2) \in [n]\times[n]\times[2]$ does not satisfy the condition of Proposition 5.6. This is a contradiction. ∎

Example 5.10 Consider the case where $n = 4$. Then the degree of the hyperdeterminant of format 3, 3, 1 with respect to T_{ijk} is 24 by Proposition 5.7. Therefore, it is a polynomial of f_{04}, f_{13}, ..., f_{40} of degree 6. By Example 4.9 (3), $\mathrm{SL}(4,\mathbb{C}) \times \mathrm{SL}(4,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$-invariant of degree 6 with respect to f_{04}, f_{13}, ..., f_{40} is a linear combination of $(f_{22}^2 - 3f_{13}f_{31} + 12f_{04}f_{40})^3$ and $(2f_{22}^3 + 27f_{04}f_{31}^2 + 27f_{13}^2 f_{40} - 9f_{13}f_{22}f_{31} - 72f_{04}f_{22}f_{40})^2$. Therefore, we see that $4(f_{22}^2 - 3f_{13}f_{31} + 12f_{04}f_{40})^3 - (2f_{22}^3 + 27f_{04}f_{31}^2 + 27f_{13}^2 f_{40} - 9f_{13}f_{22}f_{31} - 72f_{04}f_{22}f_{40})^2$ is the hyperdeterminant of format 3, 3, 1, since by Corollary 5.9, term with f_{22}^6 must vanish.

Remark 5.11 If $n = m + 1$, then the determinant (3.4) is the hyperdeterminant of format $m - 1$, m, 1 [GKZ1, Examples 4.9 and 4.10].

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